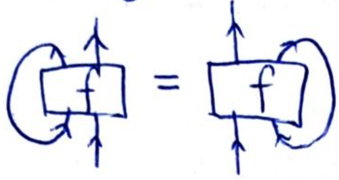
1/ April 15, 2024 (MON)

Preparing for my lecture tomorrow

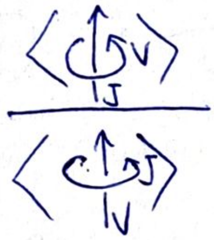
- Recall:

ambidextrous object $J \in Ob(\mathcal{C})$

(i.e.  , $\forall f \in End(J \otimes J)$)

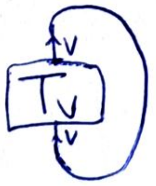
$A(J) := \{ V \in Ob(\mathcal{C})$ simple $|$  $\}$

$\Rightarrow$ <u>modified dimension</u>

$d_J(V) :=$ $d_0 \cdot$  $\in \mathbb{F}$,

<u>renormalized RT invariant</u>

$$F'(L) = d_J(V) \langle T_V \rangle \in \mathbb{F}$$

(where $L =$ )

Today: extension to 3-manifold invariants!

[Costantino-Geer-Patureau (2012)]

"Quantum invariants of 3-manifolds via link surgery presentations and non-semi-simple categories"

Let G be a (multiplicative) commutative group.

<u>Def</u> A <u>G-grading</u> in $\mathcal{C}$ is a family $\{\mathcal{C}_g\}_{g\in G}$ of full subcategories

(strict) ribbon Ab-category

such that

(1) $V \in \mathcal{C}_g,\ V' \in \mathcal{C}_{g'} \Rightarrow V \otimes V' \in \mathcal{C}_{gg'}$

(2) $V \in \mathcal{C}_g \Rightarrow V^* \in \mathcal{C}_{g^{-1}}$

(3) $V \in \mathcal{C}_g,\ V' \in \mathcal{C}_{g'},\ g \neq g \Rightarrow \mathrm{Hom}_{\mathcal{C}}(V, V') = 0$.

<u>Def</u> Let Z be an additive commutative group.

A <u>realization of Z in $\mathcal{C}$</u> is a commutative set of objects $\{\varepsilon^t\}_{t\in Z}$

braiding & twist among these objects are trivial

such that

(1) $\varepsilon^0 = \mathbb{1}$

(2) $\mathrm{qdim}(\varepsilon^t) = 1$

(3) $\varepsilon^t \otimes \varepsilon^{t'} = \varepsilon^{t+t'},\ \forall t, t' \in Z$

<u>Lemma</u> If $\{\varepsilon^t\}_{t\in Z}$ is a realization of Z in $\mathcal{C}$,

then ~~ε^t is simple,~~

$$\mathrm{Hom}_{\mathcal{C}}(V, W) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V \otimes \varepsilon^t, W \otimes \varepsilon^t) \quad \text{is an isomorphism}$$

$$f \longmapsto f \otimes \mathrm{id}_{\varepsilon^t}$$

,

and if V is simple, then $V \otimes \varepsilon^t$ is also simple.

In particular, ε^t is simple.

Given a realization of Z in $\mathcal{C}$,
we get an action of Z on isomorphism classes of objects of $\mathcal{C}$ by

$$(t, V) \longmapsto \varepsilon^t \otimes V \simeq V \otimes \varepsilon^t$$

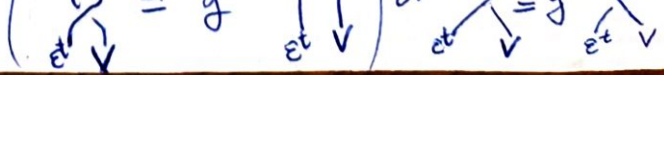
given by braiding

$\{\varepsilon^t\}_{t\in Z}$ is called a <u>free realization</u> of Z in $\mathcal{C}$
if this action is free
(i.e. if $V \otimes \varepsilon^t \not\simeq V$ for any $t \in Z \setminus \{0\}$ and $V \in \mathrm{Ob}(\mathcal{C})$)

For a simple object V, let $\tilde{V}$ be the set of isomorphism class of simple objects $\{V \otimes \varepsilon^t \mid t \in Z\}$ <u>simple Z-orbit</u>

<u>Def</u> (relative G-modular category) $\mathcal{C}$ is <u>G-modular relative to $\tilde{X}$</u> <u>with modified dimension d</u> <u>and periodicity group Z</u> if

(1) $\mathcal{C}$ has a G-grading $\{\mathcal{C}_g\}_{g\in G}$,
(2) there's a free realization $\{\varepsilon^t\}_{t\in Z}$ of Z in $\mathcal{C}_1$,
(3) there's a <u>bilinear</u> pairing $G \times Z \to K^\times$
($g^{t+t'} = g^t g^{t'}$, $(gh)^t = g^t h^t$) $(g,t) \mapsto g^t$
such that, for any $V \in \mathrm{Ob}(\mathcal{C}_g)$, $\beta_{V,\varepsilon^t} \circ \beta_{\varepsilon^t, V} = g^t \, \mathrm{id}_{\varepsilon^t \otimes V}$, $\forall t \in Z$



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(4) there exists $\tilde{X} \subset G$ such that

$\tilde{X}^{-1} = \tilde{X}$ and G cannot be covered by a finite number of translated copies of $\tilde{X}$,

(5) there is an ambidextrous pair (A, d)

(A: a set of simple objects of $\mathcal{C}$, $d: A \to \mathbb{K}^\times$, $F'(L) = d(V)\langle T_V \rangle$ is a well-defined invariant of a A-graph L)

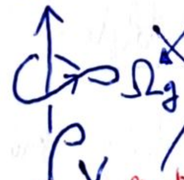
where A contains the set of simple objects of $\mathcal{C}_g$ for all $g \in G \setminus \tilde{X}$,

(6) $\forall g \in G \setminus \tilde{X}$, $\mathcal{C}_g$ is semi-simple (every object is a direct sum of simples & $\mathrm{Hom}(V, W) = 0$ for any two non-isomorphic simple objects)

and its simple objects form a union of finitely many simple $\mathbb{Z}$-orbits, finiteness!

(7) $\exists g \in G \setminus \tilde{X}$, $V \in \mathcal{C}_g$ simple

"a Kirby color of degree $g \in G$" $\Omega_g = \sum_{U \in Y} d(U)\, U$, $Y \subset \mathrm{Obj}(\mathcal{C}_g)$ finite set representing the simple $\mathbb{Z}$-orbits in $\mathcal{C}_g$

s.t. $\Delta_+ := \Big\langle$  $\Big\rangle \neq 0$

do not depend on V! follows from handle slide property

and similarly $\exists g \in G \setminus \tilde{X}$, $V \in \mathcal{C}_g$ simple

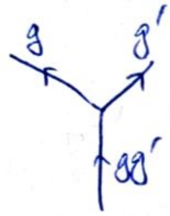
s.t. $\Delta_- := \Big\langle$  $\Big\rangle \neq 0$,

(8) $\forall V, W \in A$, $\Big\langle$  $\Big\rangle \neq 0$

A <u>homogeneous $\mathcal{C}$-color of degree $g \in G$</u> is a formal linear combination of objects in $\mathcal{C}_g$.

A <u>homogeneous $\mathcal{C}$-coloring</u> of a ribbon graph T is a coloring of each edge e of T by a homogeneous $\mathcal{C}$-color of degree $g_e \in G$, for some 1-cycle $\underbrace{\sum_e g_e[e] \in H_1(T, G)}$

$\uparrow$ G-coloring of T.



<u>Def</u> Let M be a compact connected oriented 3-manifold, T a $\mathcal{C}$-colored ribbon graph in M, and $\omega \in H^1(M \setminus T, G)$.

$$\left(\begin{array}{c} \text{injective} \\ \omega \mapsto \sum_i \omega(m_i) e_i \\ \Phi : H^1(M\setminus T, G) \to H_1(L \cup T, G) \\ \parallel \\ H_1(L,G) \oplus H_1(T,G) \end{array} \right)$$

We say (M, T, ω) is a <u>compatible triple</u> if T has a homogeneous $\mathcal{C}$-coloring given by $\Phi(\omega) \in H_1(T, G)$.

A compatible triple is <u>I-admissible</u> if there exists an edge of T colored by $\alpha \in A$.

A surgery presentation via $L \subset S^3$ for a compatible triple (M, T, ω) is <u>computable</u> if either (a) $L = \emptyset$ and $\exists$ an edge of T colored by $\alpha \in A$

or

(b) $L \neq \emptyset$ and $g_\omega(L_i) \in G \setminus \tilde{X}$ for all L_i.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \omega(m_i)$

<u>Thm</u> ([CGP 2012])

If L is a link which gives rise to a computable surgery presentation of (M,T,ω), then

$$N(M,T,\omega) = \frac{F'(L \cup T)}{\Delta_+^{p}\,\Delta_-^{s}}$$

is a well-defined topological invariant,

where (p,s) is the signature of the linking matrix of L

and each component L_i is colored by a Kirby color $\Omega_{g_\omega(L_i)}$

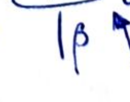
<u>Thm</u> If (M,T,ω) is T-admissible, then there exists a <u>H-stabilization</u> (M, T_H, ω_H) of (M,T,ω) admitting a computable surgery presentation.

$T \cup m$

meridian

Moreover, $$N(M,T,\omega) = \frac{F'(L \cup T_H)}{\langle H \rangle\,\Delta_+^{p}\,\Delta_-^{s}}$$

is a well-defined topological invariant of (M,T,ω),

where $\langle H \rangle := \langle$  $\rangle$

meridian added in H-stabilization

7/

~~Rmk~~ Ex

the condition (8)

Any modular category satisfying $\langle$  $\rangle \neq 0$, $\forall V, W$ simple

is an example of a relative G-modular category

with G and Z both the trivial group,

$\tilde{X} = \phi$,

A = the set of simple objects,

$d = $ qdim.

In this case, N is the usual WRT invariant.

Ex Let $G = \mathbb{C}/2\mathbb{Z}$ (additive), $\tilde{X} = \mathbb{Z}/2\mathbb{Z} \subset G$,

$Z = \mathbb{Z}$,

$\mathcal{C} :=$ tensor Ab-category of finite-dim'l weight $\overline{U}_q^H(sl_2)$-modules (ribbon), A: the set of typical modules, q: $2r$-th root of 1

Then, $\mathcal{C}$ is a relative G-modular category relative to $\tilde{X}$ with modified dimension d and periodicity group Z.

$\mathcal{C}_g :=$ the full subcat. of weight modules with weights congruent to g mod 2.

For $t \in \mathbb{Z}$, let ε^t be the 1-dim'l vector space $\mathbb{C}$ with the $\overline{U}_q^H(sl_2)$-action

$E v = F v = 0$, $K v = v$, $H v = 2rt\, v$, $\forall v \in \varepsilon^t$.

8/

$G \times \mathbb{Z} \to \mathbb{C}^\times$

$(g,t) \mapsto q^{2rt\alpha}$, where α is any complex number s.t.

$\alpha + r - 1 \equiv g \mod 2$.

For $g \in G \setminus X$ the simple modules of $\mathcal{C}_g$ are all the typical modules V_α

s.t. $\alpha + r - 1 \equiv g \mod 2$

$\varepsilon^t \otimes V_\alpha = V_\alpha \otimes \varepsilon^t = V_{\alpha + 2rt}$,

so the set of typical modules is the union of simple $\mathbb{Z}$-orbits

$\widetilde{V}_{\alpha+i}$, $i \in \{0, 1, \dots, 2r-1\}$.

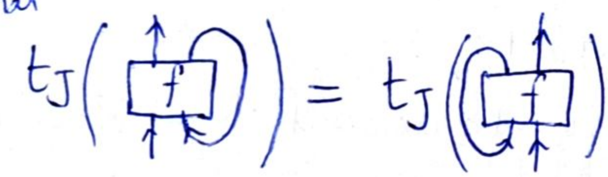
- <u>More on modified trace</u>, following [Geer-Kujawa-Patureau (2010)] "Generalized trace and modified dimension functions on ribbon categories"

<u>Def</u> For $J \in Ob(\mathcal{C})$,

an <u>ambidextrous trace on J</u> is a linear map

$$t_J : End_{\mathcal{C}}(J) \to K$$

such that



for all $f \in End(J \otimes J)$.

e.g. If J is a simple ambidextrous object, then $End_{\mathcal{C}}(J) \cong K$, and the canonical map is an ambidextrous trace.

<u>Def</u> An <u>ideal</u> I of $\mathcal{C}$ is a full subcategory s.t.

(1) ~~that~~ it is closed under retracts (i.e. $\left.\begin{matrix} W \in I \\ X \underset{\beta}{\overset{\alpha}{\rightleftarrows}} W \\ \beta\circ\alpha = Id_X \end{matrix}\right\} \Rightarrow X \in I$)

and

(2) $X \in \mathcal{C},\ Y \in I \Rightarrow X \otimes Y \in I$

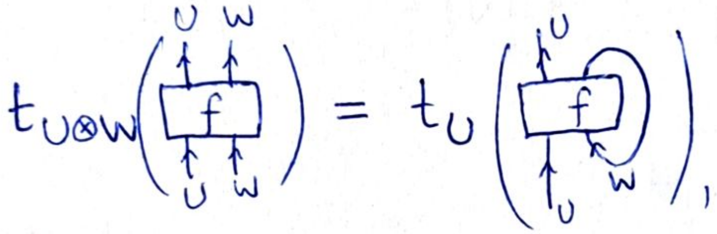
For any $J \in Ob(\mathcal{C})$, let I_J be the ideal generated by J, i.e. the ideal of all objects which are retracts of $X \otimes J$ for some X in $\mathcal{C}$.

Def A trace on an ideal I is a family of K-linear functions $t = \{t_V\}_{V \in I}$,

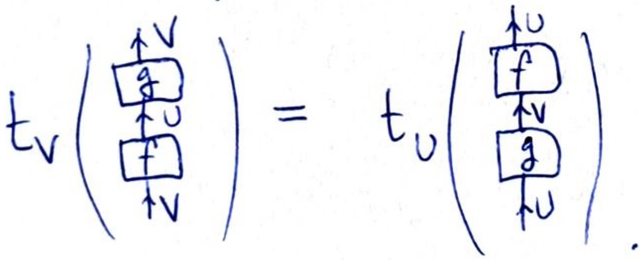
$$t_V : \mathrm{End}_{\mathcal{C}}(V) \longrightarrow K$$

such that

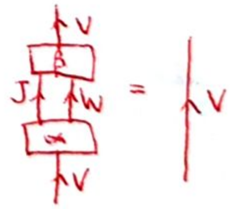
(1) $\forall\, U \in I$, $W \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{End}_{\mathcal{C}}(U \otimes W)$,



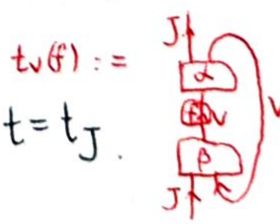
(2) $\forall\, U, V \in I$ and $f : V \to U$ and $g : U \to V$ in $\mathcal{C}$,



Thm If $\{t_V\}_{V \in I}$ is a trace on an ideal I of $\mathcal{C}$, then t_V is an ambidextrous trace for all $V \in I$.



Thm If t is an ambidexrous trace on $J \in \mathrm{Ob}(\mathcal{C})$, then there is a unique trace $\{t_V\}_{V \in I_J}$ on I_J with $t = t_J$.



Cor If V is ~~an absolutely irreducible~~ a simple ambidextrous object, then there is a unique non-zero trace on I_V up to multiplication by an element of $K^\times$.

<u>Def</u> Let $t = \{t_V\}_{V \in I}$ be a trace on an ideal I of $\mathcal{C}$.
The <u>modified dimension function</u> $d_t : Ob(I) \longrightarrow K$
is defined by $V \longmapsto t_V(id_V)$.

<u>Thm</u> Let J be an ambidextrous object in $\mathcal{C}$.
Then, $\forall V \in I_J$ with $d_J(V) \neq 0$, $I_V = I_J$.

suppose $\mathcal{C}$ is an abelian cat.
Set $Proj = \{$projective objects in $\mathcal{C}\}$ be the full sub cat of projective objects in $\mathcal{C}$.

<u>Lemma</u>

Then
- $Proj$ is an ideal in $\mathcal{C}$.
- For any ideal I of $\mathcal{C}$, $Proj \subseteq I$.
- $V \in Ob\mathcal{C}$ is projective $\Longleftrightarrow I_V = Proj$

<u>Cor</u> If J is an ambidextrous object in $\mathcal{C}$ which is not projective, then $d_J(P) = 0$ for all $P \in Proj$.

- [Costantino – Geer – Patureau (2014)]
  "Some remarks on the unrolled quantum group of sl_2"

- Simple $\overline{U}_q^H(sl_2)$-modules:

finite dim'l weight modules

Every simple module of $\mathcal{C}$ is isomorphic to exactly one of the following:

highest weight $n+kr$

- $S_n \otimes \mathbb{C}^H_{kr}$, for $n = 0, \cdots, r-2$ and $k \in \mathbb{Z}$
- V_α for $\alpha \in \ddot{\mathbb{C}} := (\mathbb{C} \setminus \mathbb{Z}) \cup r\mathbb{Z}$

"typical modules",
highest weight $\alpha + r - 1$.

$\mathbb{C}^H_{kr}$
is the 1-dim'l module
where H acts by kr

S_n is ~~a~~ (n+1)-dim'l highest weight module
with highest weight n

- For $g \in \mathbb{C}/2\mathbb{Z}$, let $\mathcal{C}_g$ be the full sub cat
  of weight modules whose weights are all in the class $g \pmod{2\mathbb{Z}}$.

<u>Thm</u>

(1) For $\bar{\alpha} \in \mathbb{C}/2\mathbb{Z} \setminus \mathbb{Z}/2\mathbb{Z}$, $\mathcal{C}_{\bar{\alpha}}$ is semisimple

(2) If $\alpha, \beta \in \ddot{\mathbb{C}}$ with $\alpha+\beta \notin \mathbb{Z}$, then $V_\alpha \otimes V_\beta \simeq \bigoplus_{k \in H_r} V_{\alpha+\beta+k}$,

where $H_r := \{-(r-1), -(r-3), \cdots, r-3, r-1\}$.

(3) All typical modules are projective.